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2005 J. Phys. A: Math. Gen. 38 11007

(http://iopscience.iop.org/0305-4470/38/50/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 11007–11017 doi:10.1088/0305-4470/38/50/009

Concurrence classes for an arbitrary multi-qubit state based on a positive operator valued measure

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Received 2 June 2005, in final form 24 October 2005
Published 30 November 2005
Online at stacks.iop.org/JPhysA/38/11007

Abstract

In this paper, we propose concurrence classes for an arbitrary multi-qubit state based on orthogonal complement of a positive operator valued measure, or POVM for short, on quantum phase. In particular, we construct concurrence for an arbitrary two-qubit state and concurrence classes for the three- and four-qubit states. Finally, we construct W^m and GHZ^m class concurrences for multi-qubit states. The unique structure of our POVM enables us to distinguish different concurrence classes for multi-qubit states.

PACS numbers: 42.50.Hz, 42.50.Dv, 42.65.Ky

1. Introduction

Entanglement is an interesting feature of quantum theory which in recent years has been attracted many researchers to quantify, classify, and investigate its useful properties. Entanglement has already some applications such as quantum teleportation and quantum key distribution, and new applications for this fascinating quantum phenomenon will surely emerge. For instance, multipartite entanglement has a capacity to offer new unimaginable applications in emerging fields of quantum information and quantum computation. One of the widely used measures of entanglement for a pair of qubits is the concurrence that gives an analytic formula for the entanglement of formation [1, 2]. In recent years, some proposals have been made to generalize this measure into a general bipartite state; e.g., Uhlmann [3] has generalized the concept of concurrence by considering arbitrary conjugation. Then Audenaert *et al* [4] generalized this formula in the spirit of Uhlmann's work, by defining a concurrence vector for the pure state. Moreover, Gerjuoy [5] and Albeverio and Fei [6] gave an explicit expression in terms of coefficients of a general pure bipartite state. Therefore, it could be interesting to try to generalize this measure from a bipartite to a multipartite system (see [7–10]). An application of concurrence for a physically realizable state such as BCS state can be found in [11]. Quantifying entanglement of multipartite states has been discussed in [12–23]. In [24, 25] we have proposed a degree of entanglement for a general pure multipartite state, based on the POVM on quantum phase. In this paper, we will define concurrence for an arbitrary two-qubit state based on the orthogonal complement of our POVM. From our POVM we will construct an operator that can be seen as a tiled operation acting on the density operator. Moreover, we will define concurrences for different classes of arbitrary three- and four-qubit states. And finally, we will generalize our result into an arbitrary multi-qubit state. The structure of our POVM enables us to detect and define different concurrence classes for multi-qubit states. The definition of concurrence is based on an analogy with bipartite states. For multi-qubit states, the W^m class concurrences are invariant under stochastic local quantum operation and classical communication (SLOCC) [20]. Furthermore, all homogeneous positive functions of pure states that are invariant under determinant-one SLOCC operations are entanglement monotones [22]. However, invariance under SLOCC for the W^m class concurrence for general multipartite states need deeper investigation. It is worth mentioning that Uhlmann [3] has shown that entanglement monotones for concurrence are related to antilinear operators. However, the GHZ^m class concurrences for multipartite states need optimization over all local unitary operations. Classification of multipartite states has been discussed in [9, 10, 26–29]. For example, Verstraete *et al* [26] have considered a single copy of a pure four-partite state of qubits and investigated its behaviour under SLOCC, which gave a classification of all different classes of pure states of four qubits. They have also shown that there exist nine families of states corresponding to nine different ways of entangling four qubits. Osterloh and Siewert [9] have constructed entanglement measures for pure states of multipartite qubit systems. The key element of their approach is an antilinear operator that they called a comb. For qubits, the combs are invariant under the action of the special linear group. They have also discussed inequivalent types of genuine four-qubit entanglement, and found three types of entanglement for these states. This result coincides with our classification, where in section 6 we construct three types of concurrence classes for four-qubit states. Miyake [27] has also discussed classification of multipartite states in entanglement classes based on the hyper-determinant. He has shown that two states belong to the same class if they are interconvertible under SLOCC. Moreover, the only paper that addresses the classification of higher-dimensional multipartite states is the paper by Miyake and Verstraete [28], where they have classified multipartite entangled states in the $2\times2\times n$ quantum systems for ($n \geqslant 4$). They have shown that there exist nine essentially different classes of states, and they give rise to a five-graded partially ordered structure, including GHZ class and W class of 3 qubits. Mintert *et al* [29] have proposed generalizations of concurrence for multipartite quantum systems that can distinguish distinct quantum correlations. However, their construction is not similar to our concurrence classes, since we can distinguish these classes based on joint phases of the orthogonal complement of our POVM by construction. Finally, Wang [10] has proposed two classes of the generalized concurrence vectors of the multipartite systems consisting of qubits. Our classification is similar to Wang's classification of the multipartite state. However, the advantage of our method is that our POVM can distinguish these concurrence classes without prior information about inequivalence of these classes under local quantum operation and classical communication (LOCC). Let us denote a general, multipartite quantum system with m subsystems by $\mathcal{Q} = \mathcal{Q}_m(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of a state $|\Psi\rangle = \sum_{k_1=1}^{N_1}\cdots\sum_{k_m=1}^{N_m}\alpha_{k_1,\ldots,k_m}|k_1,\ldots,k_m\rangle$ and $|\Psi^*\rangle = \sum_{k_1=1}^{N_1}\cdots\sum_{k_m=1}^{N_m}\alpha^*_{k_1,\ldots,k_m}|k_1,\ldots,k_m\rangle$, the complex conjugate of $|\Psi\rangle$; let $\rho_{\mathcal{Q}} = \sum_{n=1}^{\mathrm{N}} p_n|\Psi_n\rangle\langle\Psi_n|$, for all $0 \leqslant p_n \leqslant 1$, and $\sum_{n=1}^{\mathrm{N}} p_n = 1$ denotes a density operator acting on the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is given by $N_j = \dim\left(\mathcal{H}_{\mathcal{Q}_j}\right)$. We are going to use this notation throughout this paper, i.e., we denote a mixed pair of qubits by $\mathcal{Q}_2(2, 2)$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{\text{sep}}$, with respect to the Hilbert space decomposition, if it can be written as $\rho_{\mathcal{Q}}^{\text{sep}} = \sum_{n=1}^{\text{N}} p_n \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}^{n}$, $\sum_{n=1}^{\text{N}} p_n = 1$, for some positive integer N, where p_n are positive real numbers and $\rho_{\mathcal{Q}_j}^{n}$ denotes a density operator on the Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho_{\mathcal{Q}}^{p}$ represents a pure state, then the quantum system is fully separable if $\rho_{\mathcal{Q}}^{p}$ can be written as $\rho_{\mathcal{Q}}^{\text{sep}} = \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is a density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is called an entangled state. Some of the generic entangled states are called Bell states and EPR states.

2. General definition of POVM on quantum phase

In this section we will define a general POVM on quantum phase. This POVM is a set of linear operators $\Delta(\varphi_{1,2}, \ldots, \varphi_{1,N}, \varphi_{2,3}, \ldots, \varphi_{N-1,N})$ furnishing the probabilities that the measurement of a state ρ on the Hilbert space $\mathcal{H}$ is given by

$$\text{p}(\varphi_{1,2}, \ldots, \varphi_{1,N}, \varphi_{2,3}, \ldots, \varphi_{N-1,N}) = \text{Tr}(\rho\Delta(\varphi_{1,2}, \ldots, \varphi_{1,N}, \varphi_{2,3}, \ldots, \varphi_{N-1,N})), \tag{1}$$

where $(\varphi_{1,2}, \ldots, \varphi_{1,N}, \varphi_{2,3}, \ldots, \varphi_{N-1,N})$ are the outcomes of the measurement of the quantum phase, which is discrete and binary. This POVM satisfies the following properties: $\Delta(\varphi_{1,2}, \ldots, \varphi_{1,N}, \varphi_{2,3}, \ldots, \varphi_{N-1,N})$ is self-adjoint, positive and normalized, i.e.,

$$\overbrace{\int_{2\pi} \cdots \int_{2\pi}}^{N(N-1)/2} \text{d}\varphi_{1,2} \cdots \text{d}\varphi_{1,N}\, \text{d}\varphi_{2,3} \cdots \text{d}\varphi_{N-1,N}\, \Delta(\varphi_{1,2}, \ldots, \varphi_{N-1,N}) = \mathcal{I}, \tag{2}$$

where the integral extends over any 2π intervals of the form $(\varphi_k, \varphi_k + 2\pi)$ and φ_k are the reference phases for all $k = 1, 2, \ldots, N$. A general and symmetric POVM in a single N_j-dimensional Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$ is given by

$$\Delta\big(\varphi_{1_j,2_j}, \ldots, \varphi_{1_j,N_j}, \varphi_{2_j,3_j}, \ldots, \varphi_{N_j-1,N_j}\big) = \sum_{l_j}^{N_j} \sum_{k_j=1}^{N_j} \text{e}^{\text{i}\varphi_{k_j,l_j}} |k_j\rangle\langle l_j|, \tag{3}$$

where $|k_j\rangle$ and $|l_j\rangle$ are the basis vectors in $\mathcal{H}_{\mathcal{Q}_j}$ and quantum phases satisfying the following relation: $\varphi_{k_j,l_j} - \varphi_{l_j,k_j}\big(1 - \delta_{k_j l_j}\big)$. The POVM is a function of the $N_j(N_j - 1)/2$ phases $\big(\varphi_{1_j,2_j}, \ldots, \varphi_{1_j,N_j}, \varphi_{2_j,3_j}, \ldots, \varphi_{N_j-1,N_j}\big)$. It is now possible to form a POVM of a multipartite system by simply forming the tensor product

$$\Delta_{\mathcal{Q}}\big(\varphi_{\mathcal{Q}_1;k_1,l_1}, \ldots, \varphi_{\mathcal{Q}_m;k_m,l_m}\big) = \Delta_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1;k_1,l_1}\big) \otimes \cdots \otimes \Delta_{\mathcal{Q}_m}\big(\varphi_{\mathcal{Q}_m;k_m,l_m}\big), \tag{4}$$

where, e.g., $\varphi_{\mathcal{Q}_j;k_j,l_j}$ is the set of POVMs relative phase associated with subsystems $\mathcal{Q}_j$, for all $k_j, l_j = 1, 2, \ldots, N_j$, where we need only to consider when $l_j > k_j$. This POVM will play a central role in constructing concurrence classes for multi-qubit states.

3. Entanglement of formation and concurrence

In this section we will review entanglement of formation and concurrence for a pair of qubits and a general bipartite state. For a mixed quantum system $\mathcal{Q}_2(N_1, N_2)$ the entanglement of formation is defined by

$$\mathcal{E}_F(\mathcal{Q}_2(N_1, N_2)) = \inf \sum_n p_n \mathcal{E}_F\big(\rho_{\mathcal{Q}(n)}^{p}\big), \tag{5}$$

where $0 \leqslant p_n \leqslant 1$ is a probability distribution and the infimum is taken over all pure state decompositions of $\rho_{\mathcal{Q}}$. The entanglement of formation for a mixed quantum system $\mathcal{Q}_2(2,2)$ [2] can be written in terms of the Shannon entropy and concurrence as follows,

$$\mathcal{E}_F(\mathcal{Q}_2(2,2)) = \mathrm{H}\big(\tfrac{1}{2}\big(1 + (1 - \mathcal{C}^2(\mathcal{Q}_2(2,2)))^{\frac{1}{2}}\big)\big), \tag{6}$$

where $\mathcal{C}(\mathcal{Q}_2(2,2))$ is called concurrence and is defined by

$$\mathcal{C}(\mathcal{Q}_2(2,2)) = \max\left(0, \lambda_1 - \sum_{n>1} \lambda_n\right), \tag{7}$$

where, λ_n, $n = 1, \ldots, 4$ are square roots of the eigenvalues of $\rho_{\mathcal{Q}}\tilde{\rho}_{\mathcal{Q}}$ in descending order, where $\tilde{\rho}_{\mathcal{Q}}$ is given by $\tilde{\rho}_{\mathcal{Q}} = (\sigma_2 \otimes \sigma_2)\rho^*_{\mathcal{Q}}(\sigma_2 \otimes \sigma_2)$, H(X) is the Shannon entropy and $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ is the Pauli matrix. Moreover, the concurrence of a pure two-qubit, bipartite state is defined as $\mathcal{C}(\Psi) = |\langle\Psi|\widetilde{\Psi}\rangle|$, where the tilde represents the 'spin-flip' operation $|\widetilde{\Psi}\rangle = \sigma_2 \otimes \sigma_2|\Psi^*\rangle$. In the following section we will use the concept of orthogonal complement of our POVM to detect and define concurrence for an arbitrary two-qubit state and concurrence classes for arbitrary three-, four- and multi-qubit states.

4. Concurrence for an arbitrary two-qubit state

In this section we will construct concurrence for an arbitrary two-qubit state based on orthogonal complement of our POVM. For two-qubit state $\mathcal{Q}_2(2,2)$ the POVM is explicitly given by

$$\begin{aligned}\Delta_{\mathcal{Q}}\big(\varphi_{\mathcal{Q}_1;1,2}, \varphi_{\mathcal{Q}_2;1,2}\big) &= \Delta_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1;1,2}\big) \otimes \Delta_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2;1,2}\big) \\ &= \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\varphi_{\mathcal{Q}_1;1,2}} \\ \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_1;1,2}} & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\varphi_{\mathcal{Q}_2;1,2}} \\ \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_2;1,2}} & 1 \end{pmatrix}.\end{aligned} \tag{8}$$

In this POVM, the only terms that has information about joint properties of both subsystems are phase sum $\mathrm{e}^{\pm\mathrm{i}(\varphi_{\mathcal{Q}_1;1,2}+\varphi_{\mathcal{Q}_2;1,2})}$ and phase difference $\mathrm{e}^{\pm\mathrm{i}(\varphi_{\mathcal{Q}_1;1,2}-\varphi_{\mathcal{Q}_2;1,2})}$. Now, from this observation we can assume that the phase sum gives a negative contribution that is -1 and the phase difference gives a positive contribution that is +1 to a measurement. Then, we can mathematically achieve this construction by defining an operator $\widetilde{\Delta}_{\mathcal{Q}_j}\big(\varphi_{\mathcal{Q}_j;1,2}\big) = \mathcal{I}_2 - \Delta_{\mathcal{Q}_j}\big(\varphi_{\mathcal{Q}_j;1,2}\big)$, where $\mathcal{I}_2$ is a 2×2 identity matrix, for each subsystem j. Indeed by construction this operator is the orthogonal complement of our POVM. Then, we define an operator that detects entanglement as follows

$$\begin{aligned}\widetilde{\Delta}^{\mathrm{EPR}}_{\mathcal{Q}} &= \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;1,2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;1,2}\big) \\ &= \sigma_y \otimes \sigma_y,\end{aligned} \tag{9}$$

where by choosing $\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_j;k_j,l_j} = \frac{\pi}{2}$ for all $k_j < l_j$, $j = 1, 2$, we get an operator which coincides with the Pauli spin-flip operator σ_y for a single-qubit. Now, in analogy with Wootter's formula for concurrence of a quantum system $\mathcal{Q}_2(2,2)$ with the density operator $\rho_{\mathcal{Q}}$, we can define $\widetilde{\rho}^{\mathrm{EPR}}_{\mathcal{Q}}$ as

$$\widetilde{\rho}^{\mathrm{EPR}}_{\mathcal{Q}} = \widetilde{\Delta}^{\mathrm{EPR}}_{\mathcal{Q}} \rho^*_{\mathcal{Q}} \widetilde{\Delta}^{\mathrm{EPR}}_{\mathcal{Q}} \tag{10}$$

and the concurrence is given by $\mathcal{C}_\Theta(\mathcal{Q}_2(2,2)) = \max\big(0, \lambda^{\mathrm{EPR}}_1 - \sum_{n>1} \lambda^{\mathrm{EPR}}_n\big)$, where λ^{EPR}_n, $n = 1, \ldots, 4$, are square roots of the eigenvalues of $\rho_{\mathcal{Q}}\widetilde{\rho}^{\mathrm{EPR}}_{\mathcal{Q}}$ in descending order and $\rho^*_{\mathcal{Q}}$ is the complex conjugation of $\rho_{\mathcal{Q}}$. Now, we would like to extend this result to a three-qubit state.

5. Concurrence for an arbitrary three-qubit state

The procedure of defining concurrence for an arbitrary three-qubit state is more complicated than for a pair of qubits since in the three-qubit state case we have to deal with two different classes of three-partite state, namely W^3 and GHZ^3 classes. For the W^3 class, we have three types of entanglement: entanglement between subsystems one and two, $\mathcal{Q}_1\mathcal{Q}_2$, one and three, $\mathcal{Q}_1\mathcal{Q}_3$, and two and three, $\mathcal{Q}_2\mathcal{Q}_3$. So there should be three operators $\widetilde{\Delta}^{W^3}_{\mathcal{Q}_{1,2}}$, $\widetilde{\Delta}^{W^3}_{\mathcal{Q}_{1,3}}$ and $\widetilde{\Delta}^{W^3}_{\mathcal{Q}_{2,3}}$ corresponding to entanglement between these subsystems; for example, we have

$$\widetilde{\Delta}^{W^3}_{\mathcal{Q}_{1,2}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2}\big) \otimes \mathcal{I}_2, \tag{11}$$

$$\widetilde{\Delta}^{W^3}_{\mathcal{Q}_{1,3}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1}\big) \otimes \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_3}\big), \tag{12}$$

$$\widetilde{\Delta}^{W^3}_{\mathcal{Q}_{2,3}} = \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_3}\big). \tag{13}$$

Now, for a pure quantum system $\mathcal{Q}^p_3(2,2,2)$, we define the concurrence of W^3 class by

$$\begin{aligned}
\mathcal{C}\big(\mathcal{Q}^{W^3}_3(2,2,2)\big) &= \left(\mathcal{N}^W_3 \sum_{1=r_1<r_2}^{3} \big|\langle\Psi|\widetilde{\Delta}^{W^3}_{\mathcal{Q}_{r_1,r_2}}\Psi^*\rangle\big|^2\right)^{1/2} \\
&= \big(4\mathcal{N}^W_3[|\alpha_{1,2,1}\alpha_{2,1,1} - \alpha_{1,1,1}\alpha_{2,2,1} + \alpha_{1,2,2}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,2,2}|^2 \\
&\quad + |\alpha_{1,1,2}\alpha_{2,1,1} - \alpha_{1,1,1}\alpha_{2,1,2} + \alpha_{1,2,2}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,2,2}|^2 \\
&\quad + |\alpha_{1,1,2}\alpha_{1,2,1} - \alpha_{1,1,1}\alpha_{1,2,2} + \alpha_{2,1,2}\alpha_{2,2,1} - \alpha_{2,1,1}\alpha_{2,2,2}|^2]\big)^{1/2},
\end{aligned} \tag{14}$$

where $\mathcal{N}^W_3$ is a normalization constant and for a quantum system $\mathcal{Q}_3(2,2,2)$ with the density operator $\rho_{\mathcal{Q}}$, let

$$\widetilde{\rho}^{W^3}_{\mathcal{Q}} = \widetilde{\Delta}^{W^3}_{\mathcal{Q}_{r_1,r_2}} \rho^*_{\mathcal{Q}} \widetilde{\Delta}^{W^3}_{\mathcal{Q}_{r_1,r_2}}. \tag{15}$$

Then concurrence of a three-qubit mixed state of W^3 class could be defined by

$$\mathcal{C}\big(\mathcal{Q}^{W^3}_3(2,2,2)\big) = \max\left(0, \lambda^{W^3}_1(r_1,r_2) - \sum_{n>1}\lambda^{W^3}_n(r_1,r_2)\right),$$

where $\lambda^{W^3}_n(r_1,r_2)$ for all $1 \leqslant r_1 < r_2 \leqslant 3$ are square roots of the eigenvalues of $\rho_{\mathcal{Q}}\widetilde{\rho}^{W^3}_{\mathcal{Q}}$ in descending order. The second class of three-qubit state that we would like to consider is the GHZ^3 class. For the GHZ^3 class we have again three types of entanglement that give contribution to degree of entanglement, but there is a difference in construction of operators compare to the W^3 class. The operators $\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{1,2}}$, $\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{1,3}}$ and $\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{2,3}}$ that can detect entanglement between these subsystems are given by

$$\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{1,2}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\pi}_{\mathcal{Q}_3}\big), \tag{16}$$

$$\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{1,3}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\pi}_{\mathcal{Q}_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_3}\big), \tag{17}$$

$$\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{2,3}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\pi}_{\mathcal{Q}_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_3}\big), \tag{18}$$

where $\varphi^{\pi}_{\mathcal{Q}_j} = \pi$ for all j. Now, for a pure quantum system $\mathcal{Q}^p_3(2,2,2)$, we define the concurrence of GHZ^3 class by

$$\mathcal{C}\big(\mathcal{Q}^{GHZ^3}_3(2,2,2)\big) = \left(\mathcal{N}^{GHZ}_3 \sum_{1=r_1<r_2}^{3} \big|\langle\Psi|\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{r_1,r_2}}\Psi^*\rangle\big|^2\right)^{1/2}, \tag{19}$$

where $\mathcal{N}_3^{GHZ}$ is a normalization constant. For a quantum system $\mathcal{Q}_3(2,2,2)$, let $\widetilde{\rho}_{\mathcal{Q}}^{GHZ^3} = \widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{GHZ^3} \rho_{\mathcal{Q}}^* \widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{GHZ^3}$. Then concurrence of a three-qubit mixed state of GHZ^3 class is defined by

$$\mathcal{C}\big(\mathcal{Q}_3^{GHZ^3}(2,2,2)\big) = \max\left(0, \lambda_1^{GHZ^3}(r_1,r_2) - \sum_{n>1} \lambda_n^{GHZ^3}(r_1,r_2)\right), \tag{20}$$

where $\lambda_n^{GHZ^3}(r_1,r_2)$ for all $1 \leqslant r_1 < r_2 \leqslant 3$ are square roots of the eigenvalues of $\rho_{\mathcal{Q}}\widetilde{\rho}_{\mathcal{Q}}^{GHZ^3}$ in descending order. For three-qubit states the operators $\Delta_{\mathcal{Q}_{r_1,r_2}}^{W^3}$ and $\Delta_{\mathcal{Q}_{r_1,r_2}}^{GHZ^3}$ satisfy $\big(\Delta_{\mathcal{Q}_{r_1,r_2}}^{W^3}\big)^2 = 1$ and $\big(\Delta_{\mathcal{Q}_{r_1,r_2}}^{GHZ^3}\big)^2 = 1$. Now, for a state $|\Psi_{\overline{W}^3}\rangle = \alpha_{1,2,2}|1,2,2\rangle + \alpha_{2,1,2}|2,1,2\rangle + \alpha_{2,2,1}|2,2,1\rangle$, the W^3 class concurrence gives

$$\mathcal{C}\big(\mathcal{Q}_3^{W^3}(2,2,2)\big) = \big(4\mathcal{N}_3^W[|\alpha_{1,2,2}\alpha_{2,1,2}|^2 + |\alpha_{1,2,2}\alpha_{2,2,1}|^2 + |\alpha_{2,1,2}\alpha_{2,2,1}|^2]\big)^{1/2}.$$

When $\alpha_{1,2,2} = \alpha_{2,1,2} = \alpha_{2,2,1} = \frac{1}{\sqrt{3}}$, we get $\mathcal{C}\big(\mathcal{Q}_3^{W^3}(2,2,2)\big) = \big(\frac{4}{3}\mathcal{N}_3^W\big)^{1/2}$ and $\mathcal{C}\big(\mathcal{Q}_3^{GHZ^3}(2,2,2)\big) = 0$. Thus, for $\mathcal{N}_3^W = \frac{3}{4}$, we have $\mathcal{C}\big(\mathcal{Q}_3^{W^3}(2,2,2)\big) = 1$.

Moreover, let $\big|\Psi_{GHZ^3}^{\pm}\big\rangle = \alpha_{1,1,1}|1,1,1\rangle \pm \alpha_{2,2,2}|2,2,2\rangle$ and $\rho^{GHZ} = q\big|\Psi_{GHZ^3}^{+}\big\rangle\big\langle\Psi_{GHZ^3}^{+}\big| + (1-q)|\Psi_{GHZ^3}^{-}\rangle\langle\Psi_{GHZ^3}^{-}|$. Then the GHZ^3 concurrence class gives $\mathcal{C}\big(\mathcal{Q}_3^{GHZ^3}(2,2,2)\big) = \max\big(0, \lambda_1^{GHZ^3}(r_1,r_2) - \sum_{2>1}\lambda_n^{GHZ^3}(r_1,r_2)\big) = \max(0, 2q-1)$, where $\lambda_1^{GHZ^3}(1,2) = q$, $\lambda_2^{GHZ^3}(1,2) = 1-q$, and $0 < q \leqslant 1$.

As we have seen there are W^3 and GHZ^3 class concurrences for a three-qubit state. However, we are not sure how we should deal with these two different classes, but there are at least two possibilities: the first possibility is to deal with them separately, and the second one is to define an overall expression for concurrence of the three-qubit state by adding these two concurrences.

6. Concurrence classes for an arbitrary four-qubit state

In this section we will construct three different concurrences for the four-qubit states based on quantum phases of our POVM, namely the W^4, GHZ^4 and GHZ^3 class concurrences. Let us begin by constructing operators for W^4 class of four-qubit states. For the W^4 class we have six different types of entanglement: entanglement between subsystem one and two $\mathcal{Q}_1\mathcal{Q}_2$, one and three $\mathcal{Q}_1\mathcal{Q}_3$, and two and three $\mathcal{Q}_2\mathcal{Q}_3$, etc. So, there are six operators: $\widetilde{\Delta}_{\mathcal{Q}_{1,2}}^{W^4}$, $\widetilde{\Delta}_{\mathcal{Q}_{1,3}}^{W^4}$, $\widetilde{\Delta}_{\mathcal{Q}_{1,4}}^{W^4}$, $\widetilde{\Delta}_{\mathcal{Q}_{2,3}}^{W^4}$, $\widetilde{\Delta}_{\mathcal{Q}_{2,4}}^{W^4}$, $\widetilde{\Delta}_{\mathcal{Q}_{3,4}}^{W^4}$ corresponding to entanglement between these subsystems, i.e., we have

$$\widetilde{\Delta}_{\mathcal{Q}_{1,2}}^{W^4} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\frac{\pi}{2}}\big) \otimes \mathcal{I}_2 \otimes \mathcal{I}_2, \tag{21}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{1,3}}^{W^4} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\frac{\pi}{2}}\big) \otimes \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\frac{\pi}{2}}\big) \otimes \mathcal{I}_2, \tag{22}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{1,4}}^{W^4} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\frac{\pi}{2}}\big) \otimes \mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\frac{\pi}{2}}\big), \tag{23}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{2,3}}^{W^4} = \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\frac{\pi}{2}}\big) \otimes \mathcal{I}_2, \tag{24}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{2,4}}^{W^4} = \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\frac{\pi}{2}}\big) \otimes \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\frac{\pi}{2}}\big), \tag{25}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{3,4}}^{W^4} = \mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\frac{\pi}{2}}\big). \tag{26}$$

Now, for a pure quantum system $\mathcal{Q}_4^p(2,\ldots,2)$, we define the concurrence of W^4 class by

$$\mathcal{C}\big(\mathcal{Q}_4^{W^4}(2,\ldots,2)\big) = \left(\mathcal{N}_4^W \sum_{1=r_1<r_2}^{4} \big|\big\langle\Psi\big|\widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{W^4}\Psi^*\big\rangle\big|^2\right)^{1/2}, \tag{27}$$

where $\mathcal{N}_4^W$ is a normalization constant. Now, for a quantum system $\mathcal{Q}_2^{W^4}(2,\ldots,2)$ let $\widetilde{\rho}_{\mathcal{Q}}^{W^4} = \widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{W^4}\rho_{\mathcal{Q}}^*\widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{W^4}$. Then concurrence of a four-qubit mixed state of W^4 class can be defined by

$$\mathcal{C}\big(\mathcal{Q}_4^{W^4}(2,\ldots,2)\big) = \max\left(0, \lambda_1^{W^4}(r_1,r_2) - \sum_{n>1}\lambda_n^{W^4}(r_1,r_2)\right), \tag{28}$$

where $\lambda_n^{W^4}(r_1,r_2)$ for all $1 \leqslant r_1 < r_2 \leqslant 4$ are square roots of the eigenvalues of $\rho_{\mathcal{Q}}\widetilde{\rho}_{\mathcal{Q}}^{W^4}$ in descending order. The operators $\widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{W^4}$ for W^4 class satisfies $\big(\widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{W^4}\big)^2 = 1$. Now, for a state $|\Psi_{W^4}\rangle = \alpha_{1,1,1,2}|1,1,1,2\rangle + \alpha_{1,1,2,1}|1,1,2,1\rangle + \alpha_{1,2,1,1}|1,2,1,1\rangle + \alpha_{2,1,1,1}|2,1,1,1\rangle$, the W^4 class concurrence gives

$$\begin{aligned}\mathcal{C}\big(\mathcal{Q}_4^{W^4}(2,\ldots,2)\big) = \big(&4\mathcal{N}_3^W[|\alpha_{1,2,1,1}\alpha_{2,1,1,1}|^2 + |\alpha_{1,1,2,1}\alpha_{2,1,1,1}|^2 + |\alpha_{1,1,1,2}\alpha_{2,1,1,1}|^2\\ &+ |\alpha_{1,1,2,1}\alpha_{1,2,1,1}|^2 + |\alpha_{1,1,1,2}\alpha_{1,2,1,1}|^2 + |\alpha_{1,1,1,2}\alpha_{1,1,2,1}|^2]\big)^{1/2}\end{aligned}$$

and for $\alpha_{1,1,1,2} = \alpha_{1,1,2,1} = \alpha_{1,2,1,1} = \alpha_{1,2,1,1} = \frac{1}{\sqrt{4}}$, we get $\mathcal{C}\big(\mathcal{Q}_4^{W^4}(2,\ldots,2)\big) = \big(\frac{3}{2}\mathcal{N}_4^W\big)^{1/2}$, $\mathcal{C}\big(\mathcal{Q}_4^{GHZ^3}(2,\ldots,2)\big) = 0$. The second class of four-qubit state that we would like to consider is the GHZ^4 class. For GHZ^4, we have again six different types of entanglement and there are six operators defined as follows:

$$\widetilde{\Delta}_{\mathcal{Q}_{1,2}}^{GHZ^4} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\pi}\big), \tag{29}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{1,3}}^{GHZ^4} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\pi}\big), \tag{30}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{1,4}}^{GHZ^4} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\frac{\pi}{2}}\big), \tag{31}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{2,3}}^{GHZ^4} = \otimes\widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\pi}\big), \tag{32}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{2,4}}^{GHZ^4} = \otimes\widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\frac{\pi}{2}}\big), \tag{33}$$

$$\widetilde{\Delta}_{\mathcal{Q}_{3,4}}^{GHZ^4} = \otimes\widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi_{\mathcal{Q}_2}^{\pi}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi_{\mathcal{Q}_3}^{\frac{\pi}{2}}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi_{\mathcal{Q}_4}^{\frac{\pi}{2}}\big). \tag{34}$$

Now, for a pure four-qubit state $\mathcal{Q}_4^p(2,\ldots,2)$ we define the concurrence of GHZ^4 class by

$$\mathcal{C}\big(\mathcal{Q}_4^{GHZ^4}(2,\ldots,2)\big) = \left(\mathcal{N}_4^{GHZ}\sum_{1=r_1<r_2}^{4}\big|\big\langle\Psi\big|\widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{GHZ^4}\Psi^*\big\rangle\big|^2\right)^{1/2}, \tag{35}$$

where $\mathcal{N}_4^{GHZ}$ is a normalization constant and for a quantum system $\mathcal{Q}_2^{GHZ^4}(2,\ldots,2)$ with $\widetilde{\rho}_{\mathcal{Q}}^{GHZ^4} = \widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{GHZ^4}\rho_{\mathcal{Q}}^*\widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{GHZ^4}$, we define the concurrence of four-qubit mixed state of GHZ^4 class by

$$\mathcal{C}\big(\mathcal{Q}_4^{GHZ^4}(2,\ldots,2)\big) = \max\left(0, \lambda_1^{GHZ^4}(r_1,r_2) - \sum_{n>1}\lambda_n^{GHZ^4}(r_1,r_2)\right), \tag{36}$$

where $\lambda_n^{GHZ^4}(r_1,r_2)$ for all $1 \leqslant r_1 < r_2 \leqslant 4$ are square roots of the eigenvalues of $\rho_{\mathcal{Q}}\widetilde{\rho}_{\mathcal{Q}}^{GHZ^4}$ in descending order. Moreover, we have $\big(\widetilde{\Delta}_{\mathcal{Q}_{r_1,r_2}}^{GHZ^4}\big)^2 = 1$. The third class of four-qubit state that we want to consider is the GHZ^3 class. For GHZ^3, we have four different types of entanglement. So there are four operators defined as given below:

$$\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{12,3}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\pi}_{\mathcal{Q}_3}\big) \otimes \mathcal{I}_2, \tag{37}$$

$$\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{12,4}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2}\big) \otimes \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi^{\pi}_{\mathcal{Q}_4}\big), \tag{38}$$

$$\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{13,4}} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1}\big) \otimes \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_3}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi^{\pi}_{\mathcal{Q}_4}\big), \tag{39}$$

$$\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{23,4}} = \mathcal{I}_2 \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\pi}_{\mathcal{Q}_3}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_4}\big). \tag{40}$$

Then, for a pure four-qubit state $\mathcal{Q}^p_4(2,\ldots,2)$, we define concurrence for a GHZ^3 class by

$$\mathcal{C}\big(\mathcal{Q}^{GHZ^3}_4(2,\ldots,2)\big) = \left(\mathcal{N}^{GHZ}_3 \sum_{1=r_1<r_2<r_3}^{4} \big|\big\langle\Psi\big|\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{r_1r_2,r_3}}\Psi^*\big\rangle\big|^2\right)^{1/2}, \tag{41}$$

where $\mathcal{N}^{GHZ}_3$ is a normalization constant and for a quantum system $\mathcal{Q}_2(2,\ldots,2)$ with density operator $\rho_{\mathcal{Q}}$, let

$$\widetilde{\rho}^{GHZ^3}_{\mathcal{Q}} = \widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{r_1r_2,r_3}}\rho^*_{\mathcal{Q}}\widetilde{\Delta}^{GHZ^3}_{\mathcal{Q}_{r_1r_2,r_3}}. \tag{42}$$

Then concurrence for a four-qubit GHZ^3 class is defined by

$$\mathcal{C}\big(\mathcal{Q}^{GHZ^3}_4(2,\ldots,2)\big) = \max\left(0, \lambda^{GHZ^3}_1(r_1r_2,r_3) - \sum_{n>1}\lambda^{GHZ^3}_n(r_1r_2,r_3)\right), \tag{43}$$

where $\lambda^{GHZ^3}_n(r_1r_2,r_3)$ for all $1 \leqslant r_1 < r_2 < r_3 \leqslant 4$ are square roots of the eigenvalues of $\rho_{\mathcal{Q}}\widetilde{\rho}^{GHZ^3}_{\mathcal{Q}}$ in descending order. And again we have $\big(\Delta^{GHZ^3}_{\mathcal{Q}_{r_1r_2,r_3}}\big)^2 = 1$. Thus, we have detected and defined three different concurrences for the four-qubit state based on our POVM construction.

7. Concurrence classes for an arbitrary multi-qubit state

At this point, we can realize that, in principle, we could in a straightforward manner extend our construction into a multi-qubit state $\mathcal{Q}_m(2,\ldots,2)$. In order to simplify our presentation, we will use $\Lambda_m = k_1, l_1; \ldots; k_m, l_m$ as an abstract multi-index notation, where $k_j = 1, l_j = 2$ for all j. The unique structure of our POVM enables us to distinguish different classes of multipartite states, which are inequivalent under LOCC operations. In the m-partite case, the off-diagonal elements of the matrix corresponding to

$$\widetilde{\Delta}_{\mathcal{Q}}\big(\varphi_{\mathcal{Q}_1;k_1,l_1},\ldots,\varphi_{\mathcal{Q}_m;k_m,l_m}\big) = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1;k_1,l_1}\big) \otimes\cdots\otimes \widetilde{\Delta}_{\mathcal{Q}_m}\big(\varphi_{\mathcal{Q}_m;k_m,l_m}\big) \tag{44}$$

have phases that are sum or differences of phases originating from two and m subsystems. That is, in the later case the phases of $\widetilde{\Delta}_{\mathcal{Q}}\big(\varphi_{\mathcal{Q}_1;k_1,l_1},\ldots,\varphi_{\mathcal{Q}_m;k_m,l_m}\big)$ take the form $\big(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2} \pm\cdots\pm \varphi_{\mathcal{Q}_m;k_m,l_m}\big)$ and identification of these joint phases makes our classification possible. Thus, we can define linear operators for the $EPR_{\mathcal{Q}_{r_1}\mathcal{Q}_{r_2}}$ class based on our POVM which are sum and difference of phases of two subsystems, i.e., $\big(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\big)$. That is, for the $EPR_{\mathcal{Q}_{r_1}\mathcal{Q}_{r_2}}$ class we have

$$\widetilde{\Delta}^{EPR_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})} = \mathcal{I}_{2_1} \otimes\cdots\otimes \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}\big) \otimes\cdots\otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\big) \otimes\cdots\otimes \mathcal{I}_{2_m}. \tag{45}$$

Let $C(m,k) = \binom{m}{k}$ denotes the binomial coefficient. Then there are $C(m,2)$ linear operators for the $EPR_{\mathcal{Q}_{r_1}\mathcal{Q}_{r_2}}$ class and the set of these operators gives the W^m class concurrence.

For the GHZ^m class, we define the linear operators based on our POVM which are sum and difference of phases of m-subsystems, i.e., $\left(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}} \pm \cdots \pm \varphi_{\mathcal{Q}_m;k_m,l_m}\right)$. That is, for the GHZ^m class we have

$$\begin{aligned}\widetilde{\Delta}^{GHZ^m_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})} &= \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_3}}\left(\varphi^{\pi}_{\mathcal{Q}_{r_3};k_{r_3},l_{r_3}}\right) \\ &\otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_m}\left(\varphi^{\pi}_{\mathcal{Q}_{m-1};k_{r_m},l_{r_m}}\right), \end{aligned} \tag{46}$$

where by choosing $\varphi^{\pi}_{\mathcal{Q}_j;k_j,l_j} = \pi$ for all $k_j < l_j$, $j = 1, 2, \ldots, m$, we get an operator which has the structure of the Pauli operator σ_x embedded in a higher-dimensional Hilbert space and coincides with σ_x for a single-qubit. There are $C(m, 2)$ linear operators for the GHZ^m class and the set of these operators gives the GHZ^m class concurrence.

Moreover, we define the linear operators for the GHZ^{m-1} class of m-partite states based on our POVM which are sum and difference of phases of $m-1$ subsystems, i.e., $\left(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}} \pm \ldots \varphi_{\mathcal{Q}_{m-1};k_{m-1},l_{m-1}} \pm \varphi_{\mathcal{Q}_{m-1};k_{m-1},l_{m-1}}\right)$. That is, for the GHZ^{m-1} class we have

$$\begin{aligned}\widetilde{\Delta}^{GHZ^{m-1}_{\Lambda_m}}_{\mathcal{Q}_{r_1r_2,r_3}(2_{r_1},2_{r_2})} &= \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_3}}\left(\varphi^{\pi}_{\mathcal{Q}_{r_3};k_{r_3},l_{r_3}}\right) \\ &\otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{m-1}}\left(\varphi^{\pi}_{\mathcal{Q}_{m-1};k_{r_{m-1}},l_{r_{m-1}}}\right) \otimes \mathcal{I}_{2_m}, \end{aligned} \tag{47}$$

where $1 \leqslant r_1 < r_2 < \cdots < r_{m-1} < m$. There are $C(m, m-1)$ such operators for the GHZ^{m-1} class. Now, for the pure quantum system $\mathcal{Q}^p_3(2, \ldots, 2)$, we define the $EPR_{\mathcal{Q}_{r_1}\mathcal{Q}_{r_2}}$ class concurrence as

$$\mathcal{C}\left(\mathcal{Q}^{EPR_{r_1,r_2}}_m(2, \ldots, 2)\right) = \left(4\mathcal{N}^{EPR_{r_1,r_2}}_m \sum_{k_1,l_1,\ldots,;k_m,l_m} \left|\left\langle\Psi\right|\widetilde{\Delta}^{EPR_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\Psi^*\rangle\right|^2\right)^{1/2},$$

and the W^m class concurrence as

$$\mathcal{C}\left(\mathcal{Q}^{W^m}_m(2, \ldots, 2)\right) = \left(\sum_{r_2>r_1=1}^{m} \mathcal{C}^2\left(\mathcal{Q}^{EPR_{r_1,r_2}}_m(2, \ldots, 2)\right)\right)^{1/2}, \tag{48}$$

where $\mathcal{N}^{EPR_{r_1,r_2}}_m$ are normalization constants. Moreover, the GHZ^m class concurrence for general pure quantum system $\mathcal{Q}^p_m(2, \ldots, 2)$, with

$$\mathcal{C}\left(\mathcal{Q}^{GHZ^m}_{r_1,r_2}\left(2_{r_1}, 2_{r_2}\right)\right) = \sum_{\forall k_1,l_1,\ldots,k_m,l_m} \left|\left\langle\Psi\right|\widetilde{\Delta}^{GHZ^m_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\Psi^*\rangle\right|^2, \tag{49}$$

is given by

$$\mathcal{C}\left(\mathcal{Q}^{GHZ^m}_m(2, \ldots, 2)\right) = \left(\mathcal{N}^{GHZ}_m \sum_{r_2>r_1=1}^{m} \mathcal{C}\left(\mathcal{Q}^{GHZ^m}_{r_1,r_2}\left(2_{r_1}, 2_{r_2}\right)\right)\right)^{1/2}, \tag{50}$$

where $\mathcal{N}^{GHZ}_m$ is a normalization constant. Now, let us address the monotonicity of these concurrence classes of multipartite states. For m-qubit states, the W^m class concurrences are entanglement monotones. Let $A_j \in SL(2, \mathbf{C})$, for $j = 1, 2, \ldots, m$, and $\mathcal{A} = A_1 \otimes A_2 \otimes \cdots \otimes A_m$, then $\mathcal{A}\widetilde{\Delta}^{W^m_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\mathcal{A}^T = \widetilde{\Delta}^{W^m_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}$, for all $1 < r_1 < r_2 < m$. Thus, the W^m class concurrences for multi-qubit states are invariant under SLOCC, and hence are entanglement monotones. Again, for general multipartite states we cannot give any proof on invariance of W^m class concurrence under SLOCC and this question needs further investigation. Moreover, for multipartite states, the GHZ^m class concurrences are not entanglement monotones except under additional conditions. Since $\mathcal{A}\widetilde{\Delta}^{GHZ^m_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\mathcal{A}^T \neq$

$\widetilde{\Delta}^{GHZ^3_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}$, for all $1 < r_1 < r_2 < m$. The reason is that $A_j\widetilde{\Delta}_{\mathcal{Q}_j}\big(\varphi^{\pi}_{\mathcal{Q}_j;1,2}\big)A^T_j \neq \widetilde{\Delta}_{\mathcal{Q}_j}\big(\varphi^{\pi}_{\mathcal{Q}_j;1,2}\big)$. Thus, the GHZ^m class concurrence for three-qubit states are not invariant under SLOCC, and hence are not entanglement monotones. However, by construction the GHZ^m class concurrences are invariant under all permutations. Moreover, we have $\big(\widetilde{\Delta}^{GHZ^3_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\big)^2 = 1$ and $\big(\widetilde{\Delta}_{\mathcal{Q}_j}\big(\varphi^{\pi}_{\mathcal{Q}_j;1,2}\big)\big)^2 = 1$. Furthermore, we need to be very careful when we are using the GHZ^m class concurrences. This class can be zero even for an entangled multipartite state. Thus, for the GHZ^m class concurrences we need to perform an optimization over local unitary operations. For example, let $\mathcal{U} = U_1 \otimes U_2 \otimes \cdots \otimes U_m$, where $U_j \in U(2, \mathbf{C})$. Then we maximize the GHZ^m class concurrences for a given pure m-partite state over all local unitary operations $\mathcal{U}$.

Finally, e.g., for the W^m class for a general quantum system $\mathcal{Q}_m(2, \ldots, 2)$ with the density operator $\rho_{\mathcal{Q}}$, we define

$$\widetilde{\rho}^{W^m_{\Lambda_m}}_{\mathcal{Q}} = \widetilde{\Delta}^{EPR_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\rho^*_{\mathcal{Q}}\widetilde{\Delta}^{EPR_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})} \tag{51}$$

and then the W^m class concurrence is defined by

$$\mathcal{C}\big(\mathcal{Q}^{W^m_{\Lambda_m}}_2(2, \ldots, 2)\big) = \max\left(0, \lambda^{W^m_{\Lambda_m}}_1(r_1, r_2) - \sum_{n>1}\lambda^{W^m_{\Lambda_m}}_n(r_1, r_2)\right), \tag{52}$$

where $\lambda^{W^m_{\Lambda_m}}_n(r_1, r_2)$ for all $1 \leqslant r_1 < r_2 \leqslant m$ are the square roots of the eigenvalues of $\rho_{\mathcal{Q}}\widetilde{\rho}^{W^m_{\Lambda_m}}_{\mathcal{Q}}$ in descending order. The GHZ^m class concurrences for a quantum system $\mathcal{Q}_m(N2, \ldots, 2)$ can be defined in a similar way. The definition of concurrence classes for multipartite mixed states is only a well-motivated suggestion and is a generalization of Wootters and Uhlmann definitions. Moreover, our operators $\Delta^{X_m}_{\mathcal{Q}_{r_1,r_2}}$ satisfy $\big(\Delta^{X_m}_{\mathcal{Q}_{r_1,r_2}}\big)^2 = 1$. As an example of a multi-qubit state let us consider a state $|W^m\rangle = \frac{1}{\sqrt{m}}(|1, 1, \ldots, 1, 2\rangle + \cdots + |2, 1, \ldots, 1, 1\rangle)$. For this state the W^m class concurrence is

$$\mathcal{C}\left(\mathcal{Q}^{W^m}_m(2, \ldots, 2)\right) = \left(\frac{4C(m, 2)}{m^2}\mathcal{N}^W_m\right)^{1/2} = \left(\frac{2(m-1)}{m}\mathcal{N}^W_m\right)^{1/2}. \tag{53}$$

This value coincides with the one given by Dür [20]. Finally, for some partially separable states the $\mathcal{C}\big(\mathcal{Q}^{W^m}_m(2, \ldots, 2)\big)$ class and $\mathcal{C}\big(\mathcal{Q}^{GHZ^m}_m(2, \ldots, 2)\big)$ class concurrences do not exactly quantify entanglement in general. Example of such states can be for example, constructed for three-qubit states. Thus, we may need to define a overall concurrence by adding these concurrence classes.

8. Conclusion

In this paper we have expressed concurrence for an arbitrary two-qubit state, based on our POVM, which coincides with the Wootters original formula. Moreover, we have generalized this result into arbitrary three- and four-qubit states. For three-qubit states, we have found two different concurrence classes and for four-qubit states, we have constructed three concurrence classes. Finally, we have generalized our result into arbitrary multi-qubit state and we have explicitly constructed W^m and GHZ^m class concurrences. We have investigate the monotonicity of the W^m class and the GHZ^m class concurrences for multi-qubit states. The W^m class concurrence for multi-qubit states are entanglement monotones. However, GHZ^m class concurrences need optimization over all local unitary operation. Our construction suggested the existence of different classes of multipartite entanglement which are inequivalent under LOCC. At least, we know that there are two different classes of entanglement for multi-qubit states which our methods could distinguish very well. But we can also define an overall expression for concurrence with a suitable normalization coefficient. However, we think that this work is a timely contribution to the relatively large effort presently being undertaken to quantify and classify multipartite entanglement.

Acknowledgments

The author acknowledges useful discussions with Gunnar Björk and Piotr Badziag. The author also would like to thank Jan Bogdanski. This work was supported by the Wenner–Gren Foundations.

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